E=mc² for dummies

For a particle that we accelerate from a speed 0 up to a speed v applies

$$E_k = \int F\,dx = \int \frac{dp}{dt}\,dx = \int \frac{dp}{dt}\,v\,dt = \int v\,dp$$

For relativistic speeds applies $p = \frac{m}{\sqrt{1-(\frac{v}{c})^2}}.v$ (see Wikipedia)

m is the so-called rest mass.

We replace $\frac{v}{\sqrt{1-(\frac{v}{c})^2}}$ with x

=> $1/x^2 = 1/v^2 - 1/c^2$ so $1/v^2 = 1/x^2 + 1/c^2$ => $v = c.\frac{x}{\sqrt{c^2+x^2}}$

The above integral can now rewritten as:

$$E_k = mc\int_0^x \frac{x}{\sqrt{c^2+x^2}}\,dx = m\,c\left[\sqrt{c^2+x^2}\right]_0^x$$

Finding the primitive function of the integral is mainly a matter of trying. The more you do it the faster you find the solution. You can also solve the integral with the Wolfram online integrator (see here)

Between the borders 0 and x :

$$E_k = mc\sqrt{c^2+x^2} - mc^2$$

It says here that the kinetic energy is equal to the total energy

$E_{tot} = mc\sqrt{c^2+x^2}$ **minus** the energy at rest ($v=0$ so $x=0$)

so $E_{rust} = mc\sqrt{c^2+0} = mc^2$ ($E = mc^2$)

From $x = \frac{v}{\sqrt{1-(\frac{v}{c})^2}}$ follows that $v = \frac{x}{\sqrt{1+(\frac{x}{c})^2}}$ or $v = \frac{1}{\sqrt{1/x^2+1/c^2}}$ from which follows that

$\sqrt{1+x^2/c^2} = \frac{1}{\sqrt{1-v^2/c^2}}$ see: **Calculating with relativistic quantities can be a lot easier.**

The equation E_k rewritten for v :

$$E_k = mc\sqrt{c^2+x^2} - mc^2 = mc^2\sqrt{1+x^2/c^2} - mc^2 = mc^2\left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right)$$ (Wikipedia)